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A Method of Characteristics and Invariant Imbedding for Distributed Control Problems*

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A method of characteristics and invariant imbedding is introduced for solving optimal control problems for hyperbolic systems. Advantages over existing methods are illustrated by means of examples.

1. INTRODUCTION

In this paper, we will consider those control problems where the external inputs to a distributed system are the control functions. These controls are to be determined in order to satisfy a criterion functional. Using standard techniques [2, 3, 11], determination of the solution of such a problem requires knowledge of the Green's function. Determination of a Green's function necessitates solving of an associated Sturm–Liouville eigenvalue problem. This is usually a difficult numerical problem to solve on a digital computer. Alternatively, a direct method using characteristics and invariant imbedding [1] is shown here for solving certain optimal control problems in distributed parameter systems.

2. HYPERBOLIC SYSTEMS

Let a system be described by a vector set of partial differential equations,

$$\frac{\partial u}{\partial t}(x, t) + A(u, x, t)\frac{\partial u}{\partial x}(x, t) + b(u, x, t) = 0, \tag{1}$$

where u is an n-dimensional state vector, A is an $m \times m$ square matrix, b is

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an m vector, $x \in [0, L]$, and $t \geqslant 0$. Let λ_i, $i = 1, \ldots m$, be m roots of the characteristic equation of the operator A, given by,

$$| A - \lambda I | = 0, \tag{2}$$

where I is the identity matrix and $| A |$ means the determinant of A. If all the characteristic roots $\lambda_1, \ldots \lambda_m$ of the matrix A are real, and if, in addition, the matrix A has a full set of m linearly independent, real, characteristic vectors $V_1, \ldots V_m$, then the system (1) is described as being hyperbolic. The curves in the t, x plane, described by the differential equation,

$$\frac{dx}{dt} = \lambda_i(x, t), \qquad i = 1, \ldots, m, \tag{3}$$

are called characteristic curves or simply, characteristics. These curves have many properties, which may be found in [4, 5].

3. A Distributed Control Problem

Let the system be described by the following first-order partial differential equation,

$$\frac{\partial u}{\partial t}(x, t) + \frac{\partial u}{\partial x}(x, t) = - u(x, t) + m(t), \tag{1}$$

with the initial conditions at $t = 0$ and $x = 0$ as,

$$u(x, 0) = f(x), \qquad x > 0, \tag{2}$$

$$u(0, t) = 0, \qquad t > 0. \tag{3}$$

The function $m(t)$ is the control to be determined such that,

$$J = \int_0^T \int_0^L u^2(x, t)\, dx\, dt + \rho \int_0^T m^2(t)\, dt, \tag{4}$$

is minimum.

Using the calculus of variations [1], the necessary conditions for a minimum are found to be

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = v - 2u, \qquad 0 < x < L, \quad 0 < t < T, \tag{5}$$

$$v(x, T) = 0, \qquad 0 < x < L \tag{6}$$

$$v(L, t) = 0, \qquad 0 < t < T, \tag{7}$$

$$m(t) = -\frac{1}{2\rho} \int_0^L v(x, t)\, dx, \qquad 0 \leqslant t \leqslant T. \tag{8}$$

Substituting for $m(t)$ in Eq. (1) we get,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = - u - \frac{1}{2\rho} \int_0^L v(x, t)\, dx. \tag{9}$$

Equations (5) and (9) are partial differential equations with two point boundary values in both the variables x and t, given by Eqs. (2), (3), (6), and (7).

4. Initial-Value Methods

The above example has been considered in [6]. There, the author has tried to solve the two-point boundary-value problem by reducing it to an initial value problem in the "steady state", via the following transformation,

$$\int_0^L v(x, t)\, dx = \int_0^L K(x, t)\, u(x, t)\, dx. \tag{1}$$

This transformation does not lead to an initial value problem for all $t > 0$ [6]. However, we can reduce the two-point boundary-value problem to an initial-value problem via the following linear transformation.

$$v(x, t) = \int_0^L R(x, x', t)\, u(x', t)\, dx', \qquad 0 \leqslant x \leqslant L. \tag{2}$$

Thus, we seek a linear relationship between u and v. This is justified since u and v are related by linear functional equations [1, 7]. Related work on initial-value methods for boundary-value problems may be found in [8, 9, 10].

Differentiating (2) with respect to t, we get

$$\frac{\partial v}{\partial t}(x, t) = \int_0^L \frac{\partial R}{\partial t}(x, x', t)\, u(x', t)\, dx' + \int_0^L R(x, x', t) \frac{\partial u}{\partial t}(x', t)\, dx'. \tag{3}$$

Using (3.5) and (3.9) we get

$$\begin{aligned} -\frac{\partial v}{\partial x} + v - 2u = \int_0^L \frac{\partial R}{\partial t}(x, x', t)\, u(x', t)\, dx' + \int_0^L R(x, x', t) \\ \times \left(-\frac{\partial u}{\partial x'}(x', t) - u(x', t) - \frac{1}{2\rho} \int_0^L v(x'', t)\, dx'' \right) dx'. \end{aligned} \tag{4}$$

Differentiating (2) with respect to x, we get

$$\frac{\partial v}{\partial x}(x, t) = \int_0^L \frac{\partial R}{\partial x}(x, x', t)\, u(x', t)\, dx'. \tag{5}$$

Using (2) and (5) in (4) and rearranging terms we get

$$\int_0^L \left(-\frac{\partial R}{\partial x}(x, x', t) + 2R(x, x', t) - \frac{\partial R}{\partial t}(x, x', t)\right) u(x', t)\, dx' - 2u(x, t)$$

$$= - \int_0^L R(x, x', t) \frac{\partial u}{\partial x'}(x', t)\, dx' - \frac{1}{2\rho} \int_0^L R(x, y, t)\, dy \int_0^L \int_0^L R(x'', x', t)$$

$$\times\, u(x', t)\, dx'\, dx''.$$

Integrating the first term on the right-hand side by parts, using (3.3), and collecting terms in u we obtain

$$\int_0^L \left(-\frac{\partial R}{\partial x}(x, x', t) - \frac{\partial R}{\partial t}(x, x', t) + 2R(x, x', t) - \frac{\partial R}{\partial x'}(x, x', t)\right.$$

$$\left. + \frac{1}{2\rho} \int_0^L R(x, y, t)\, dy \int_0^L R(x'', x', t)\, dx'' - 2\delta(x' - x)\right) u(x', t)\, dx'$$

$$= - R(x, L, t)\, u(L, t), \tag{6}$$

where $\delta(x)$ is the Dirac delta function.

Impose the condition,

$$R(x, L, t) = 0. \tag{7}$$

Since (6) must hold for all u, we must have,

$$\begin{aligned} &\frac{\partial R}{\partial x}(x, x', t) + \frac{\partial R}{\partial x'}(x, x', t) + \frac{\partial R}{\partial t}(x, x', t) - 2R(x, x', t) \\ &\quad - \frac{1}{2\rho} \int_0^L R(x, y, t)\, dy \int_0^L R(x'', x', t)\, dx'' + 2\delta(x' - x) = 0. \end{aligned} \tag{8}$$

Using (3.6) and (3.7) in (2) we get

$$R(x, x'T) = 0, \qquad 0 \leqslant (x, x') \leqslant L, \tag{9}$$

and

$$R(L, x', t) = 0, \qquad 0 \leqslant x' \leqslant L, \qquad 0 \leqslant t \leqslant T. \tag{10}$$

Equation (8) can be solved backwards as an initial-value problem subject to the initial condition (9) and the auxiliary conditions (7) and (10). Knowing R, we can solve for u, and m forwards, via

$$m(t) = - \frac{1}{2\rho} \int_0^L \int_0^L R(x, x', t)\, u(x', t)\, dx'\, dx \tag{11}$$

and

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = - u + m, \qquad u(0, t) = 0, \qquad u(x, 0) = f(x).$$

This is called a two sweep method (since we have to solve equations forwards and backwards). A detailed discussion on such methods is available in [7].

Equations (6)–(11) could also be obtained directly by using a Dynamic Programming approach [12, 1].

5. Method of Characteristics and Invariant Imbedding for Certain Distributed Control Problems

Let the given system equation be

$$\frac{\partial u}{\partial t}(x, t) + \frac{\partial u}{\partial x}(x, t) = - u(x, t) + f(x, t), \tag{1}$$

subject to

$$u(x, 0) = h(x), \qquad x > 0 \tag{2}$$

and

$$u(0, t) = g(t), \qquad t > 0. \tag{3}$$

It is required to determine the control function $f(x, t)$, such that

$$J = \tfrac{1}{2} \int_0^T \int_0^L u^2 \, dx \, dt + \tfrac{1}{2} \int_0^T \int_0^L f^2 \, dx \, dt \tag{4}$$

is minimum.

Using the calculus of variations, one can show that the necessary conditions for a minimum are

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = v - u, \tag{5}$$

$$f \equiv - v, \tag{6}$$

$$v(x, T) = 0, \qquad x < L, \tag{7}$$

and

$$v(L, t) = 0, \qquad t < T. \tag{8}$$

Substitution of (6) in (1) yields,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = - u - v. \tag{9}$$

Equations (5) and (9) together with the conditions (2), (3), (7), and (8) form a two-point boundary-value problem in both the variables x and t. The standard technique of solving such equations is by the method of lines. The partial differential equations are either reduced to matrix difference equations in two variables or matrix ordinary differential equations in one variable by discretizing the other variable. Invariant imbedding techniques are used to solve these matrix difference (or differential equations) as initial-value problems in one sweep [9, 10]. In the sequel, we will show a method of characteristics and invariant imbedding to solve the problem as a one-sweep method without relying on any approximations. Also, for any point (x_1, t_1) in the (x, t) plane, the solution is determined by solving, simply, three scalar equations. Thus, the dimensionality problem associated with solving matrix equations is obviated. The method presented can be easily extended to systems described by vector equations.

Equations (5) and (9) form a hyperbolic system whose characteristics are given by

$$\frac{dx}{dt} = 1. \tag{10}$$

Along the characteristics (5) and (9) become

$$\frac{du}{dt}(x, t) = -u(x, t) - v(x, t), \tag{11}$$

$$\frac{dv}{dt}(x, t) = v(x, t) - u(x, t). \tag{12}$$

Let (x_1, t_1) be a fixed point in the x, t plane. Along $dx/dt = 1$, we have

$$x_1 - x = t_1 - t$$

or

$$x = x_1 - t_1 + t \tag{13}$$

and

$$t = t_1 - x_1 + x. \tag{14}$$

Using (13) in (11) and (12) we have

$$\frac{du}{dt}(x_1 - t_1 + t, t) = -u(x_1 - t_1 + t, t) - v(x_1 - t_1 + t, t), \tag{15}$$

$$\frac{dv}{dt}(x_1 - t_1 + t, t) = v(x_1 - t_1 + t, t) - u(x_1 - t_1 + t, t). \tag{16}$$

Conditions (2), (3), (7), and (8) become

$$u(x_1 - t_1, 0) = h(x_1 - t_1), \qquad x_1 > t_1, \tag{17}$$

$$u(0, t_1 - x_1) = g(t_1 - x_1), \qquad x_1 < t_1, \tag{18}$$

$$v(x_1 - t_1 + T, t) = 0, \qquad x_1 - t_1 + T < L,$$

and,

$$v(L, t_1 - x_1 + L) = 0, \qquad t_1 - x_1 + L < T.$$

The equations (15) and (16) form a linear ordinary two-point boundary-value problem, whose boundary conditions are known once we choose x_1 and t_1 (see Fig. 1). For example, let the point (x_1, t_1) be such that

$$x_1 > t_1 \quad \text{and} \quad t_1 - x_1 + L < T.$$

Call

$$x_1 - t_1 = a, \qquad t_1 - x_1 + L = L - a \triangleq b,$$

$$u(x_1 - t_1 + t, t) = u(a + t, t) = U_1(t), \tag{19}$$

$$v(x_1 - t_1 + t, t) = v(a + t, t) = V_1(t). \tag{20}$$

Equations (15) and (16) become

$$\frac{dU_1}{dt} = -U_1 - V_1, \tag{21}$$

$$\frac{dV_1}{dt} = V_1 - U_1, \tag{22}$$

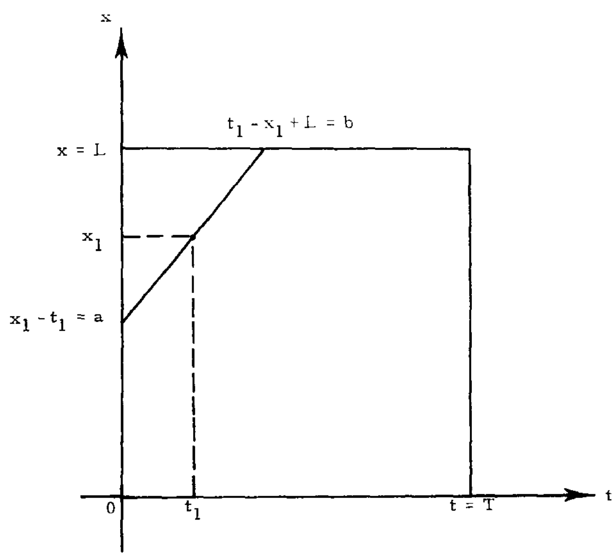

FIG. 1. The characteristics through (t_1, x_1).

subject to

$$U_1(0) = h(a), \tag{23}$$

$$V_1(b) = 0. \tag{24}$$

From (19) and (20),

$$u(x_1, t_1) = U_1(t_1) \qquad \text{and} \qquad v(x_1, t_1) = V_1(t_1).$$

Now consider a more general problem, where

$$\frac{dU}{dt} = -U - V, \qquad \frac{dV}{dt} = V - U,$$

subject to

$$U(c) = 1, \qquad V(b) = 0.$$

In the invariant imbedding technique, we consider a family of U and V with different starting instants "c". Therefore,

$$U = U(t, c), \qquad V = V(t, c).$$

Thus, we can write

$$\dot{U}(t, c) = -U(t, c) - V(t, c), \tag{25}$$

$$\dot{V}(t, c) = V(t, c) - U(t, c), \tag{26}$$

$$U(c, c) = 1, \qquad V(b, c) = 0, \tag{27}$$

where the dot means differentiation with respect to the variable t. Clearly U_1 and V_1 are given by

$$U_1(t) = h(a)\, U(t, 0), \qquad \text{or} \qquad u(x_1, t_1) = U_1(t_1) = h(a)\, U(t_1, 0), \tag{28}$$

$$V_1(t) = h(a)\, V(t, 0), \qquad \text{or} \qquad v(x_1, t_1) = V_1(t_1) = h(a)\, V(t_1, 0). \tag{29}$$

From (25), (26), and (27), $U_c(t, c)$ and $V_c(t, c)$ satisfy

$$\dot{U}_c = -U_c - V_c, \tag{30}$$

$$\dot{V}_c = V_c - U_c, \tag{31}$$

subject to

$$\begin{gathered} \dot{U}(c, c) + U_c(c, c) = 0, \qquad \text{i.e.,} \qquad U_c(c, c) = -\dot{U}(c, c), \\ V_c(b, c) = 0, \end{gathered} \tag{32}$$

where subscript c means partial differentiation with respect to c. Comparison of Eqs. (30)–(32) with (25)–(27) gives, as a consequence of linearity, the following relationship,

$$U_c(t, c) = -\dot{U}(c, c)\, U(t, c),$$
$$V_c(t, c) = -\dot{U}(c, c)\, V(t, c).$$

Using (25) and (27) we have

$$\dot{U}(c, c) = -1 - V(c, c).$$

Therefore,

$$U_c(t, c) = (1 + V(c, c))\, U(t, c), \tag{33}$$
$$V_c(t, c) = (1 + V(c, c))\, V(t, c). \tag{34}$$

Define

$$r(c) = V(c, c).$$

Therefore,

$$\frac{dr}{dc} = \dot{V}(c, c) + V_c(c, c).$$

Using (26), (27), and (34) we get

$$\frac{dr}{dc} = (V(c, c) - 1) + (1 + V(c, c))\, V(c, c),$$

or

$$\frac{dr}{dc} = r^2 + 2r - 1. \tag{35}$$

The initial condition on r is given at $c = b$, as

$$r(b) = V(b, b) = 0. \tag{36}$$

Starting from $c = b$, we integrate (35) backwards up to $c = t$, where t is fixed. At $c = t$, we adjoin equations (33) and (34), whose initial conditions are given as

$$U(t, t) = 1, \qquad V(t, t) = r(t).$$

Integration is carried out until $c = 0$ is reached. Thus, we obtain $U(t, 0)$ and $V(t, 0)$ for any fixed t (say t_1) in one sweep. Therefore, from (28) and (29) we have

$$u(x_1, t_1) = h(a)\, U(t_1, 0)$$

and

$$v(x_1, t_1) \equiv f(x_1, t_1) = h(a)\, V(t_1, 0).$$

6. Discussion

It was shown that invariant imbedding can be used along the characteristics to solve certain optimal distributed control problems for hyperbolic systems. The solution is obtained in terms of only three initial-value, ordinary differential equations, in one sweep, and without relying on any discretizing approximations. Thus, besides obtaining a one-sweep method which reduces storage requirements on the computer and makes it possible to synthesize on-line controllers, we have also achieved a reduction in dimensionality. Extension of this method to systems with many families of characteristics needs further investigation.

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